Reps of quivers over the virtual field $\mathbb{F}_1$

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Plan: §1 Philosophy of $\mathbb{F}_1$

§2. Quiver reps over $\mathbb{F}_1$ (after Szczensy)

§3. Homological properties of rep$(Q, \mathbb{F}_1)$

§1. Philosophy of $\mathbb{F}_1$.

$\mathbb{F}_1$ = "the field of characteristic one" = Virtual field.

Tits (1956): Γ be a geometry, $\lim_{q\to 1} \Gamma(\mathbb{F}_q)$ should be a geometry defined over $\mathbb{F}_1$.

Manin (1995): translating the geometric proof of the Weil Conjectures from function field to $\mathbb{Q}$

$\rightsquigarrow$ Algebraic geometry over $\mathbb{F}_1$.

Exam 1. $\mathbb{F}_1$ - vector space.

Let V be an n-dim. vector space.

$$V(\mathbb{F}_q) = \mathbb{F}_q \varepsilon_1 \oplus \cdots \oplus \mathbb{F}_q(\varepsilon_n), \quad \varepsilon_1, \ldots \varepsilon_n \text{ a basis of } V$$

$$\lim_{q\to 1} V(\mathbb{F}_q) = ?$$

Def. • An $\mathbb{F}_1$-vector space is a pointed set $V = (V, 0_V)$.

$\dim V = |V| - 1$ is the dimension of V.

Exam 2. $Gr(k, n)$: k-dimensional subspaces in an n-dim space

$$\lim_{q\to 1} Gr(k, n)(\mathbb{F}_q) = \{k\text{- subsets of the set of } n \text{ elements}\}$$

$$|Gr(k, n)(\mathbb{F}_q)| = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

$$\lim_{q\to 1} |Gr(k, n)(\mathbb{F}_q)| = \binom{n}{k}.$$

Def • Let V, W be $\mathbb{F}_1$- vector spaces. An $\mathbb{F}_1$- linear map $f: V \to W$ from V to W is a map s.t

$$\begin{cases} f(0_V) = 0_W \\ f|_{V \setminus f^{-1}(0_W)} \text{ is an injection.} \end{cases}$$

• Denote by $\mathrm{Hom}(V, W)$ the set of all the $\mathbb{F}_1$- linear maps.

$\mathrm{Vect}(\mathbb{F}_1)$ the cat. of $\mathbb{F}_1$- vector spaces.

Rk. • $\mathrm{Hom}(V, W)$ has no additive structure but only a pointed set with

$$\begin{aligned} 0: V &\to W \\ \alpha &\mapsto 0_W \end{aligned}$$

• $\mathrm{Vect}(\mathbb{F}_1)$ is not additive, but has almost all the good properties as the one $\mathrm{Vect}(k)$.

In particular, we have kernel, cokernel, direct sum, ...

§2. Quiver representation over $\mathbb{F}_1$.

<u>Def</u>. Let $Q = (Q_0, Q_1)$ be a finite quiver.

The reps of Q over $\mathbb{F}_1$ can be defined as usual.

i.e. $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ M_i : vector space$/\mathbb{F}_1$

M_α : $\mathbb{F}_1$-linear map.

$$\begin{tikzcd} M: \arrow[d, "f"] & M_i \arrow[r, "M_\alpha"] \arrow[d, "f_i"] \arrow[dr, phantom, "\circlearrowleft"] & M_j \arrow[d, "f_j"] \\ N: & N_i \arrow[r, "N_\alpha"] & N_j \end{tikzcd}$$

$\forall \alpha \in Q_1$

$f = (f_i)_{i \in Q_0}$ is a morphism from M to N.

- $\underline{\dim} M = (\dim M_i)_{i \in Q_0}$ dimension vector of M
- $\dim M = \sum_{i \in Q_0} \dim M_i$ dimension of M

Denote by $\mathrm{rep}(Q, \mathbb{F}_1)$ the cat. of f.d. $\mathbb{F}_1$-reps of Q.

Rk. • $\mathrm{rep}(Q, \mathbb{F}_1)$ has many good properties as the one $\mathrm{rep}(Q, k)$

i.e. $\mathrm{rep}(Q, \mathbb{F}_1)$ is a proto-exact category, i.e. a non-additive analogue of Quillen's exact category

$\rightsquigarrow$ Hall alg theory.

Thm (Szczesny 2012) (If Q has oriented cycles. Consider the nilpotent reps)

- Jordan-Hölder Thm for $\mathrm{rep}(Q, \mathbb{F}_1)$
- Krull-Schmidt Thm for $\mathrm{rep}(Q, \mathbb{F}_1)$

Thm (Szczesny 2012)

Let Q be a connected tree quiver.

$$\{\text{indec. reps of } Q\}/\sim \;\longleftrightarrow\; \{\text{connected subquiver of } Q\}$$

$$M(Q') \longleftarrow\!\!\shortmid \; Q'$$

$$M(Q')_i = \begin{cases} \mathbb{F}_1 & i \in Q' \\ 0 & \text{else} \end{cases}$$

$$M(Q')_\alpha = \begin{cases} \mathrm{id}_{\mathbb{F}_1} & \alpha \in Q' \\ 0 & \text{else.} \end{cases}$$

Rk: A connected quiver is of rep-finite $\Leftrightarrow$ tree quiver.
[Jun-Sistko 23]

§3. Homological properties of $\mathrm{rep}(Q, \mathbb{F}_1)$

For $\mathrm{rep}(Q, k)$, the Euler form

$$\langle L, M \rangle = \sum_{i \geq 0} (-1)^i \dim \mathrm{Ext}^i(L, M)$$

play an important role in the study of reps of Q and also in Ringel's realization of positive part of quantum enveloping alg.

☆ For $\mathrm{rep}(Q, \mathbb{F}_1)$, Yoneda's construction can be applied to define $\mathrm{Ext}^i(L, N)$ for $L, N \in \mathrm{rep}(Q, \mathbb{F}_1)$, $i \in \mathbb{N}$.

- $\mathrm{gl.dim}\, \mathrm{rep}(Q, \mathbb{F}_1) \triangleq \sup\left\{ t \;\middle|\; \begin{array}{l} \exists L, N \in \mathrm{rep}(Q, \mathbb{F}_1) \\ \mathrm{Ext}^t(L, N) \neq 0 \end{array} \right\}$

- $\mathrm{rep}(Q, \mathbb{F}_1)$ is hereditary $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{gl.dim}\, \mathrm{rep}(Q, \mathbb{F}_1) \leq 1$.

Szczesny proposed the following expectation:

(a) Is $\mathrm{rep}(Q, \mathbb{F}_1)$ hereditary?

(b) Is the Euler form $\langle \, , \rangle$ well-defined?

(c) If the Euler form $\langle \, , \rangle$ is well-defined, does it descend to

$$G_0(\mathrm{rep}(Q, \mathbb{F}_1)) \cong \mathbb{Z}^{|Q_0|} ?$$

Thm (F - Ran - Yang 2023)

Let Q be a linear quiver of type A_n.

$$\mathrm{gl.dim}\ \mathrm{rep}(Q, \mathbb{F}_1) \leq 1 \iff n \leq 2$$

$$\mathrm{gl.dim}\ \mathrm{rep}(Q, \mathbb{F}_1) = 2 \iff n \geq 3$$

Cor $\langle -, - \rangle$ is well-define for $\mathrm{rep}(Q, \mathbb{F}_1)$ for linear quiver Q of type A_n.

(show $\mathrm{Ext}^i(L, M)$ is a finite pointed set)

<u>key idea</u>: · indec. reps of linear quiver are uniserial!

· A splitting lemma: $M \overset{f}{\twoheadrightarrow} N_0 \oplus N_1$

$\Rightarrow \quad M = M_0 \oplus M_1 \qquad M_0 \oplus M_1 \xrightarrow{\begin{bmatrix} f_0 & 0 \\ 0 & f_1 \end{bmatrix}} N_0 \oplus N_1$

f_0, & f_1 surjective!

<u>ex</u>. $Q: 2 \to 1$. $(\Rightarrow \text{gl.dim}\, \text{rep}(Q, \mathbb{F}_1) \leq 1)$

$S_1 : 0 \to \mathbb{F}_1 \qquad S_2 : \mathbb{F}_1 \to 0. \quad P_2 : \mathbb{F}_1 \xrightarrow{1_{\mathbb{F}}} \mathbb{F}_1.$

⤷ projective obj

$\forall\, L, M \in \text{rep}(Q, \mathbb{F}_1)$

$$\langle L, M \rangle = \dim \text{Hom}(L, M) - \dim \text{Ext}^1(L, M)$$

$\langle P_2 \oplus P_2, P_2 \rangle = \dim \text{Hom}(P_2 \oplus P_2, P_2) = 2.$

$\underline{\dim}\, P_2 \oplus P_2 = \dim S_1 \oplus S_2 \oplus P_2, \qquad \dim P_2 = \dim S_1 \oplus S_2$

$\left.\begin{array}{l} \dim \text{Hom}(S_1 \oplus S_2 \oplus P_2, S_1 \oplus S_2) = ? \ (5) \\ \dim \text{Ext}^1(S_1 \oplus S_2 \oplus P_2, S_1 \oplus S_2) = 1 \end{array}\right\} \Rightarrow \langle S_1 \oplus S_2 \oplus P_2, S_1 \oplus S_2 \rangle = 4$

$\rightsquigarrow \langle -, - \rangle$ does not descend to $G_0(\text{rep}(Q, \mathbb{F}_1))$.

Rk One can define projective objs as usual.

- In general; $\mathrm{Ext}^1(L,-)=0 \not\Rightarrow L$ is projective.

$Q: \ 1 \longrightarrow 2 \longleftarrow 3$

$L: \ \mathbb{F}_1 \xrightarrow{I} \mathbb{F}_1 \xleftarrow{I} \mathbb{F}_1$

$\mathrm{Ext}^1(L,-)=0$

& L is not proj.